Teaching and learning of mathematics and criteria for its improvement from the perspective of future teachers: a view from the Ontosemiotic Approach

Adriana Breda[1], María José Seckel[2], Danyal Farsani[3], José Fernandes da Silva[4], Eulalia Calle[5]

[1]Universitat de Barcelona, Spain, [2]Universidad Católica del Maule, Chile, [3]Universidad Finis Terrae, Chile, [4]Instituto Federal de Minas Gerais, Brazil, [5]Universidad de Cuenca, Ecuador

adriana.breda@ub.edu, mjseckel@gmail.com, dfarsani@uft.cl, jose.fernandes@ifmg.edu.br, eulalia.calle@ucuenca.edu.ec

Abstract: *The objective of this article is to identify the meaning attributed to the didactics of mathematics and what are the criteria with which an improvement in the teaching and learning process of mathematics is based, future teachers of mathematics, belonging to universities in three different countries (Brazil, Chile, and Ecuador). The qualitative analysis indicates that the majority of future teachers consider that the didactics of mathematics is a technical discipline that consists of providing strategies, resources, and procedures for teaching mathematics; few consider it as an art to teaching and almost none consider it as a scientific discipline that is concerned with studying the processes of teaching and learning mathematics. In addition, the results show that the criteria used by them, on how teaching and learning in this discipline can be improved, are focus, above all, in the cognitive, ecological, and emotional aspects and, to a lesser extent, to the interactional, mediational and epistemic. Finally, it is concluded that the improvement in teaching and learning is directly related to an improvement in the training programs of future mathematics teachers.*

INTRODUCTION

There are different characteristics that the training and professional development programs of mathematics teachers must possess. One of the topics of discussion in mathematics didactics is directly or indirectly related to improving teaching and learning processes. A goal of these teacher training programs is to achieve an impact on the improvement of mathematics teaching. This aspect generates the need to understand the following questions: What is understood by the process of improving the teaching and learning of mathematics? And what is the role of mathematics didactics in supporting this process?

There are different improvements that can be considered in the area of mathematics didactics. The first can be perceived as training programs. A second way is to follow certain trends in the teaching of mathematics, considering that this can generate improvements (Bishop, Clements, Keitel, Kilpatrick, and Leung, 2003; English, Bartolini-Busi, Jones, Lesh, and Tirosh, 2008; Gutiérrez and Boero, 2006; Lester, 2007). On the other hand, a third way is to follow some principles and patterns that guide the development of the practice of mathematics teaching (NCTM, 2000; Breda, Font, and Pino-Fan, 2018). These principles were and are generated by the community interested in mathematics education. In particular, the community of researchers in the Didactics of Mathematics and its scientific contributions, and also the educational community in general (teachers, students, parents, administration, etc.).

Some studies have addressed the role of Mathematics Didactics (MD), focusing on understanding the conceptions about MD in trainers of future teachers and the development of didactic-mathematical skills or abilities in future teachers (Oliveira and Fiorentini, 2018; Nortes and Nortes, 2011; Rosa, Farsani, and Silva, 2020), others have developed research focused on the beliefs and conceptions of future teachers about mathematics education and the teaching and learning processes of that discipline (Gvozdic and Sander, 2018; Marbán, Palacios and Maroto, 2020; Manderfeld and Siller, 2019) and others have focused their interest on the meaning given to MD from the teaching experiences experienced by classroom teachers (Zumaeta, Fuster and Ocaña, 2018; Farsani, 2015; 2016). Also, there are researchers who have been concerned with studying how MD is constituted as a research field. Some of the conclusions they have reached is that the Didactics of Mathematics is constituted in the middle of power strategies. It operates in different ways and is not subordinate to other disciplines (Fernandes, 2014).

In line with the aforementioned, this article aims to identify the meaning that future mathematics teachers from three Latin American countries (Brazil, Chile, and Ecuador) attribute to the didactics of mathematics and the criteria with which they base an improvement in the process of teaching and learning mathematics.

In the following sections we present: a) The theoretical framework, which explains the different understandings about what MD is and its role as a discipline, the idea of improvement in the teaching and learning of mathematics and, finally, the construct Didactical Suitability Criteria (DSC) of the Ontosemiotic Approach (OSA); b) The methodology, which presents the context of the study, the instrument with which the data were collected and the procedure for analysing them; c) The results and discussions; d) The considerations of this investigation.

LITERATURE REVIEW

In this section, we will present a brief description of what MD is and its research from the perspective of different theorists in the area, as well as what we consider to be an improvement in the teaching and learning processes of mathematics. We finish by explaining the construct that we have chosen as a tool to prescribe, evaluate and analyze the teaching and learning processes of the discipline under study, the DSC.

Didactics of Mathematics

Since its inception (Steiner, 1985; Brousseau, 1989), research on the teaching of mathematics was shaped by the field of educational research, which consequently changed its initial focus of philosophical speculations. The didactics of mathematics is an art of teaching, which is a process used by teachers helping their students develop mathematical skills and new knowledge (Kilpatrick, 1998). In particular, the evolution of MD, as a result of its direct relationship with changes in research on mathematical knowledge, has led it to try to characterize itself as a scientific discipline (Gascón, 1998; Gascón and Nicolás, 2017). Steiner (1985, p.11) observed that:

Mathematics Education can never become a science or a field with scientific foundations. The field is then left open for highly subjective views and beliefs, for short range pragmatism and an interpretation of mathematics teaching as primarily an art.

The scientific character, as Brousseau (1989) points out, is classified into a) the applied multidisciplinary conception - serves to instruct the necessary teachings for the professional training of teachers and as a field of research carried out on teaching considering the scientific disciplines such as psychology, pedagogy, sociology, semiotics, etc. b) the autonomous conception - fundamental of the discipline itself, mathematics.

The concepts presented lead us to understand MD as a scientific or technical discipline, endowed with methodological aspects. These methodological aspects serve to explain how the teaching and learning processes of mathematics are carried out without referring to prescriptive aspects or evaluative.

On the other hand, Schoenfeld (2000) at the beginning of the 21st century, when proposing his questions concerning the nature of research in the didactics of mathematics, argues that it has two main purposes, one pure and the other applied. For this author, the pure is related, above all, in understanding the nature of mathematical thinking, teaching, and learning, while the applied purpose is related, above all, in using that understanding to improve mathematical instruction. Godino (2006; 2010) corroborates the idea of Schoenfeld (2000) and the idea that MD is a scientific discipline made up of three large fields: a) Didactic technology, b) Scientific research and, c) Practical and reflective action. Didactic technology aims to develop materials and resources, using available scientific knowledge. Scientific research tries to understand the functioning of the teaching of mathematics as a whole. Finally, practical and reflective action examines the teaching and learning processes of mathematics. For this author, these three fields are focused on the operation of didactic systems and have an ultimate purpose: the improvement of the teaching and learning of mathematics. With the same purpose, Lesh and Sriramn (2010), consider MD as science-oriented to the design of processes and resources to improve the teaching and learning processes of mathematics.

Godino, Batanero, and Font (2019) consider that MD has a scientific and technological character, thus demonstrating a broad conception of MD as a scientific discipline, since it must consider that it must address theoretical issues of mathematical knowledge itself (its ontological characteristics, epistemological, semiotic), descriptive, explanatory, predictive questions (relationships of the theoretical questions of mathematical knowledge with the teaching and learning processes), typical of scientific knowledge, and also prescriptive and evaluative questions, typical of technological knowledge.

The ideas presented lead us to classify the different understandings related to MD. The first is to understand MD as an art of teaching, which takes away the scientific character of the discipline. The second is to understand MD as a scientific discipline that is based on methods and theories that help us describe and explain how mathematical knowledge is generated and how its teaching and learning processes are developed. Finally, the third is to understand MD as science-oriented towards the improvement of the teaching and learning processes of mathematics, that is, its prescriptive and evaluative characteristics.

Improvement in the teaching of mathematics: philosophical aspects

Considering the MD perspective that leans towards prescriptive and/or evaluative aspects, in this section, what is understood by an improvement of mathematics teaching and learning processes is clarified.

The idea of improving teaching and learning processes is related, above all, to the idea of truth. That is, the truth about what is considered as process improvement can come, from philosophy, starting from a positivist or consensual perspective (Nicolás, 1997).

From the positivist perspective, what is correct, incorrect, good, bad, has quality (or not) will be told by the progress of this scientific area, which will find objective results that will guide us to improve the teaching and learning processes. At its core, it is a positivist discourse based on the theory of truth as correspondence (Font and Godino, 2011). From this point of view, the strategy to improve the teaching and learning processes of mathematics should be of the top/bottom type. Now, the main problem with this way of understanding change and improvement is that teachers are not included in the process, they are limited to applying curricular materials designed by experts dedicated to research. This perspective, while giving great importance to the role of theory, limits the role of the teacher to that of the user and does not take much into account the socio-political factors that affect mathematics education.

The consensual perspective is a notion inspired by the idea of Peirce's consensual theory of truth and its developments by Apel and Habermas (Nicolás, 1997). From this perspective, the Didactics of Mathematics can offer us provisional principles agreed by the interested community, which can serve to guide and assess the teaching and learning processes of mathematics. As explained in Breda, Font, and Pino-Fan (2018), trends in mathematics teaching are a first way of observing consensus in the Educational Mathematics community, since they can be considered as regularities found in the speeches on the improvement of the teaching of mathematics (Guzmán, 2007). A second way of observing consensus is the reconversion of some of these trends into explicit principles, such as the principles and standards of the National Council of Teachers of Mathematics (NCTM, 2000), which emerge from a broad consensus among teachers, the association of teachers, trainers of mathematics teachers, representatives of the educational administrations, researchers and mathematicians, all of them with extensive educational experience.

In addition to that, in the Didactics of Mathematics knowledge and results have been generated that enjoy wide consensus. Along these lines, a characteristic of many theoretical approaches to the area is that, in addition to assuming some principles for the development of their theoretical construction, they consider that these principles should be considered in the teaching of mathematics so that it is better.

For the development of the didactical suitability construct, current trends on the teaching of mathematics, the NCTM principles, and the contributions of the different theoretical approaches in the area of Mathematics Didactics have been considered (Godino, 2013; Breda, Font, and Pino-Fan, 2018).

In the OSA, the didactical suitability criteria (DSC), its components, and characteristics were constructed on the basis that they should be constructs that rely on a certain amount of consensus within the Mathematics Education community. As a result, it was considered that, given the ample consensus they generate, the principles of the NCTM (2000) could serve as the basis for some of the DSC, or rather, they could be considered as some of the components themselves. On the other hand, for the development of the didactical suitability construct, some of the contributions (principles, results, etc.) of the different approaches of the Mathematics Education area were also taken into account (Godino, 2013). Therefore, one of the plausible explanations that the suitability criteria can be considered as teachers' reflection patterns is related to the extensive consensus that they generate amongst persons involved in Mathematics Education (Breda, Pino-Fan and Font, 2017).

Didactical Suitability Criteria

The DSC arises in response to the following question: What kind of actions and resources should be implemented in the instructional processes to optimize mathematical learning? According to the OSA, the notion of didactical suitability is defined as the degree to which the process (or a part of it) meets certain characteristics that allow it to be classified as optimal or adequate to achieve adaptation between the personal meanings achieved by the students (learning) and the institutional meanings intended or implemented (teaching), considering the circumstances and available resources (environment) (Godino, Batanero and Font, 2019). As we pointed out previously, DSC can serve, a priori, to guide (or plan) the teaching and learning processes of mathematics and, in the aftermath, to assess their implementations. The OSA considers the following DSC (Font, Planas, and Godino, 2010):

1. Epistemic suitability, to assess whether the mathematics being taught is "good mathematics".

2. Cognitive suitability, to assess, before starting the instructional process, if what is to be taught is at a reasonable distance from what the students know, and after the process, if the acquired learning is close to what was intended to teach.

3. Interactional suitability, to assess whether the interactions resolve doubts and difficulties of the students.

4. Mediational suitability, to assess the adequacy of the material and temporal resources used in the instructional process.

5. Emotional Suitability, to assess the involvement (interests and motivations) of students during the instructional process.

6. Ecological Suitability, to assess the adequacy of the instructional process to the educational project of the center, to the curricular guidelines, and the conditions of the social and professional environment.

The operation of the DSC requires defining a set of observable indicators, which allow assessing the degree of suitability of each of these criteria. For example, there is consensus that it is necessary to implement "good" mathematics, but it is possible to understand very different things for it. For some DSC, the indicators are relatively easy to agree on (for example, for the criterion of the suitability of means), for others, as is the case of epistemic suitability, it is more difficult. Once the six criteria of partial suitability have been determined, each one of them is broken down into components and indicators, which make them operational. Tables 1, 2, 3, 4, 5, and 6 present the DSC, published in Breda, Pino-Fan, and Font (2017).

Components	Characteristics
Errors	✓ Practices considered mathematically incorrect are not observed.
Ambiguities	✓ Ambiguities that could confuse students are not observed; definitions and procedures are clear and correctly expressed, and adapted to the target level of education; explanations, evidence and demonstrations are suitable for the target level of education, the use of metaphors is controlled, etc.
Diversity of processes	✓ Relevant processes in mathematical activity (modelling, argumentation, problem-solving, connections, etc.) are considered in the sequence of tasks.
Representation	✓ The partial meanings (constituted of definitions, properties, procedures, etc.), are representative samples of the complexity of the mathematical notion chosen to be taught as part of the curriculum. ✓ For one or more partial meanings, a representative sample of problems is provided. ✓ The use of different modes of expression (verbal, graphic, symbolic…), treatments and conversations amongst students are part of one or more of the constituents of partial sense.

Table 1: Components and characteristics of epistemic suitability.

Components	Characteristics
Previous knowledge (similar components to those of epistemic suitability)	✓ Students have the necessary previous knowledge to study the topic (that is, they have previously studied or the teacher makes a study plan). ✓ The intended meanings (reasonable difficulty) can be taught through its diverse components.
Adaptation of the curriculum to the individuals' different needs	✓ Development and support activities are included.

Learning	✓ The diverse methods of evaluation demonstrate the application of intended or implemented knowledge/competences.
High cognitive demand	✓ Relevant cognitive processes are activated (generalization, intra-mathematical connections, changes of representations, speculations, etc.) ✓ Metacognitive processes are promoted.

Table 2: Components and characteristics of cognitive suitability.

Components	Characteristics
Teacher-student interaction	✓ The teacher appropriately presents the topic (clear and well-organized presentation, not speaking too fast, emphasis on the key concept of the topic, etc.) ✓ Students' conflicts of sense are recognized and resolved (students' silence, facial expressions, questions are correctly interpreted and an appropriate survey is conducted, etc.) ✓ The aim is to reach a consensus on the best argument. ✓ Varieties of rhetorical and rational devices are used to involve the students and capture their attention. ✓ The inclusion of students into the class dynamic is facilitated –exclusion is not.
Interaction amongst learners	✓ Dialogue and communication between students is encouraged. Inclusion in the group is preferred and exclusion is discouraged.
Autonomy	✓ Moments in which students take on responsibility for their study (exploration, formulation and validation) are observed.
Formative evaluation	✓ Systematic observation of the cognitive progress of the students.

Table 3: Components and characteristics of interactional suitability.

Components	Characteristics
Material resources (manipulatives, calculators, computers)	✓ The use of manipulatives and technology, which give way to favorable conditions, language, procedures, and arguments, adapted to the intended sense. ✓ Definitions and properties are contextualized and motivated using concrete situations, models, and visualizations.
Number of students, scheduling, classroom conditions	✓ The number and distribution of students enables the desired teaching to take place. ✓ The timetable of the course is appropriate (for example, not all the classes are held late) ✓ The classroom and the distribution of the students is appropriate for the development of the intended instructional method.
Time (for group teaching/tutorials; time for learning)	✓ Accommodating the intended/implemented content to the available time (contact or non-contact hours) ✓ Devotion of time to the most important or central aspects of the topic. ✓ Devotion of time to topic areas that present more difficulty.

Table 4: Components and characteristics of mediational suitability.

Components	Characteristics
Interests and needs	The selection of tasks that are of interest to the students. Introduction of scenarios that enable students to evaluate the practicality of mathematics in everyday situations and professional life.
Attitudes	Promoting involvement in activities, perseverance, responsibility, etc. Reasoning should be done so in a context of equality; the argument will be valued in its own right and not by the person who puts it forward.
Emotions	Promotion of self-esteem, avoiding rejection, phobia or fear of mathematics. Aesthetic qualities and the precision of mathematics are emphasized.

Table 5: Components and characteristics of affective suitability.

Components	Characteristics

Adaptation to the curriculum	✓	The content, its implementation and evaluation, correspond to the curricular plan.
Intra/Interdisciplinary connections	✓	The content is related to other mathematical topics (connection of advanced mathematics with curricular mathematics and the connection between different mathematics content covered in the curriculum) or to the content of other disciplines, (an extra-mathematical context or rather links with other subjects from the same educational stage).
Social-professional practicality	✓	The course content is useful for socio-professional insertion.
Didactical Innovation	✓	Innovation based on reflexive research and practice (introduction of new content, technological resources, methods of evaluation, classroom organization, etc.)

Table 6: Components and characteristics of ecological suitability.

As mentioned, both the components and the indicators of the DSC have been made considering the consensual perspective of the truth, that is, the trends, principles, and results of research in the area of Didactics of Mathematics. Particularly, for epistemic suitability, this principle has been considered. Mathematical objects emerge from practices, which embodies their complexity (Font, Godino and Gallardo, 2013; Rondero and Font, 2015). From this principle, a component is derived (representativeness of complexity) whose objective is to consider the mathematical complexity in the design and redesign of the didactic sequences (Pino-Fan, Castro, Godino, and Font, 2013; Monje, Seckel, and Breda, 2018).

METHOD

To identify the meaning they attribute to mathematics didactics and the criteria with which they base an improvement in the mathematics teaching process, a qualitative research methodology has been used (Lüdke and André, 1986). In what follows, we explain the research context, the data collection instrument, and the data analysis process.

Research context

This study was conducted with forty-nine students (future teachers) who were studying Basic General Education with a major and a Bachelor's degree in Mathematics in three Latin American countries: Brazil, Chile, and Ecuador. The students of the Basic General Education program with a mention belong to a Chilean subsidized university and an Ecuadorian public university. For their part, the students of the Bachelor's degree in Mathematics belong to a Brazilian public university.

The twenty-four future professors of the Ecuadorian public university, located in the region of Azogues in southern Ecuador, were studying the fifth semester (approximately in the middle of the course) of the Basic General Pedagogy career - which presents a total of nine semesters and medium. In this career, the choice to mention in a specific discipline (mathematics or language) is available from the seventh semester. In this sense, the participants were taking, for the first time, an introductory mathematics didactic course called Teaching and Learning Mathematics I, whose objective was to know, analyze and design strategies and resources in the area of mathematics. It is important to emphasize that, until the fifth semester, the students had not taken any specific subject in mathematics or mathematics didactics, but they had taken pedagogy and pre-professional practices from the first semester of the program or career.

The seventeen future professors from the Brazilian public university, located in the Minas Gerais region in south-eastern Brazil, had already completed nearly 50% of the Bachelor's degree in Mathematics - which has a total of eight semesters. The students had already taken the disciplines of Pedagogical Practices in the Teaching of Mathematics, General Didactics, Teaching of 'Greatness and Measurement, Teaching of Geometry', Teaching of Statistics and Financial Mathematics, Computational Resources, Laboratory of Teaching of Mathematics I, and Supervised Practice I. Also, they had taken specific subjects in mathematics (geometry, algebra, and calculus). Of the seventeen participants in this research, fifteen participated in an introductory teaching project promoted by the Teaching Initiation Scholarship Program (or PIBID in *Programa de Bolsas de Iniciação à Docencia*).

The group of eight future professors from the Chilean subsidized university (both public and private), located in the Maule region in central-south Chile, were studying the seventh semester of the study plan of the Basic General Pedagogy career with a mention in Mathematics - which presents a total of ten semesters. Until the seventh semester, the study participants had already taken five mathematics subjects, each with a didactic approach. It should be noted that the training plan for these participants includes internships in schools from the fourth semester.

Data collection

The data were collected through a questionnaire of four following open questions: a) What do you understand by Didactics of Mathematics? b) What kind of questions should the Didactics of Mathematics answer? c) What does an improvement in the teaching of mathematics mean? and d) How to develop didactic proposals that represent an improvement in the teaching and learning of mathematics? In particular, the first two questions of the instrument tried to find information about what future teachers understood by Didactics of Mathematics and the last two tried to collect information about the meaning of improvement in the teaching and learning of mathematics attributed by them.

Data analysis

To analyze the participants' discourse and identify what they understand by Mathematics Didactics, we have considered the a priori categories presented in the section on Mathematics Didactics present in the theoretical framework. That is, the responses of the future teachers were categorized considering the different understandings of MD present in the literature: the art of teaching, scientific discipline and/or technological discipline.

On the other hand, to analyze the discourse of the participants when they refer to what is an improvement in the teaching and learning of mathematics and how one can develop didactic proposals that represent an improvement concerning the teaching of mathematics that is usually carried out, a priori categories have also been used. The categories that we have considered are the didactical suitability criteria proposed by the OSA (epistemic, mediational, ecological, emotional, interactional, and cognitive suitability) and its components and indicators. The reason for the use of taking DSC as categories of analysis comes from the constitution of the construct itself (Breda, Font and Pino-Fan, 2018), in particular, it is based on the implicit or explicit assumption that there are certain trends related to the mathematics teaching that indicate how an improvement in mathematics teaching should be compared to what is usually done (Font, 2008; Guzmán, 2007). In particular, it was sought to categorize according to the DSC, which are the criteria assumed by future teachers when they declare how the improvement of mathematics teaching should be carried out.

RESULTS AND DISCUSSION

When analyzing the responses of the forty-nine future teachers about what meaning they attribute to MD, we have observed that twenty-eight of them understand MD as a technological discipline, that is, as a set of techniques, procedures, and resources that serve to improve the teaching and learning of mathematics. For example, as evidenced in the following units of analysis:

The didactics of mathematics are the methods in which teachers develop the attitudes, skills, and knowledge of students for their teaching and learning (Student 4, Chile).

It is a set of actions, applied by teachers in the classroom, which serves to instruct students in mathematical knowledge (Student 17, Brazil).

I understand that mathematics didactics refers to how and with which the teaching and learning of mathematics can be improved and students can understand the contents more easily. I also understand that these are strategies that can be applied within the classroom that respond to the students' needs (Student 1, Ecuador).

Three understand it as an art of teaching, not considering the scientific or technological aspect of the discipline:

Didactics is related to how mathematics can be taught, that is, how we students teach subjects related to mathematics. Didactics is an art and in mathematics, it is an art of reaching students understandably. (Student 5, Ecuador).

For their part, three future teachers understand it as a scientific discipline that is concerned, in addition to the techniques, procedures, and resources, of studying the teaching and learning processes of mathematics:

MD is a set of teaching and learning in conjunction with pedagogy and psychology. It is understood as the relationship between theory and practice in which it leads the student to think, create and build (Student 10, Brazil).

The didactics of mathematics refers to the study of the teaching and learning processes of mathematical sciences. (Student 6, Ecuador).

The results obtained in this first topic allow us to observe that the responses are very similar in the three countries that have participated in the research. In other words, of the seventeen participants from the Brazilian university, fourteen assume that MD is a technical discipline, two consider it the art of teaching and one considers it a scientific discipline. On the other hand, of the twenty-four participants from the University of Ecuador, nine have answered this question, where it is observed that seven consider MD as a technical discipline, one considers it the art of teaching and one considers it a scientific discipline. Finally, of the eight students at the Chilean university, seven consider MD to be a technical discipline and one considers it a scientific discipline. In sum, most future teachers assume in their discourse the technological character of MD, that is, they understand MD as science-oriented towards the improvement of the teaching and learning processes of mathematics.

Regarding the criteria with which future teachers base the improvement of the teaching and learning process of mathematics, it is evident that they implicitly consider the use of the six criteria of didactical suitability.

Regarding epistemic suitability, a future Ecuadorian professor argues that an improvement in the teaching and learning of mathematics consists in working on the richness component of mathematical processes, in particular, the problem-solving process, which is evidenced below:

Identify the situations (problem). Investigate the existence of similar (previous) problems, develop possible solutions and evaluate them, prepare proposals (Student 7, Ecuador).

Regarding cognitive suitability, future teachers considered the importance of working on students' prior knowledge. For them, the improvement consists of:

Being able to learn by teaching, because it is not enough just to fill out the blackboard and believe that the students learned, without at least knowing that the students already know something (Student 4, Brazil).

Diagnose the academic level of students, find the shortcomings they have had in previous years and establish flexible strategies. (Student 18, Ecuador).

Investigating where are the deficiencies in the learning of mathematics and being able to reinforce and also identify what type of content is relevant (Student 6, Chile).

In addition to having an overview related to prior knowledge, improvement consists of making curricular adaptations to individual differences, another component of cognitive suitability.

Emphasize personalized teaching considering the virtues of children. (Student 17, Ecuador).

They have also considered the importance of activating relevant cognitive processes, an indicator of the high cognitive demand component of the cognitive suitability criterion.

It must be considered that as a teacher, students are not looking for the correct result, but rather that they learn to reason and that they arrive at the correct answer. That is, take into consideration the process and not just the product. (Student 22, Ecuador).

Future teachers considered it important to consider the teacher-student interaction, a component of the interactional criterion:

It means an advance in the communication processes between the parties involved in learning so that the process occurs effectively. (Student 15, Brazil).

From an emotional point of view, future teachers declare that improving mathematics teaching and learning consists in promoting self-esteem, avoiding rejection, phobia, or fear of mathematics, which is related to the component called "emotions" of the emotional suitability criteria (Breda, Pino-Fan and Font, 2017):

An improvement means overcoming the prejudice about mathematics and the use of pedagogical and technological resources so that the student learns without trauma or fear and becomes a subject who learns with confidence. (Student 8, Brazil).

They also consider that it is important to propose situations of interest to students, having the "interests and needs" component of the criterion of interactional suitability implicit:

Transmit mathematics in a clear and simple way so that students do not get bored, or lose interest, much less come to hate mathematics (Student 3, Ecuador).

Related to the media, many students considered that in order to have an improvement in the teaching and learning of mathematics, it is important to use diverse and adequate resources a resource component of the mediational suitability criterion.

It means a positive change in the strategies and resources that are used to teach mathematics (Student 7, Ecuador).

Depending on the age of the students and their needs, recreational activities can be carried out and specific material used. (Student 3, Ecuador).

They also considered it important that the didactic proposals take into account the socio-labor utility, a component of the ecological suitability criterion:

I believe that when inserted in the reality of the student, in a more practical way it would improve the teaching of mathematics. (Teacher 9, Brazil).

Change the paradigm and the conception that we have of mathematics and its teaching. Working mathematics from a more contextualized perspective (...) the main thing that we find useful for what is learned in this subject in everyday life. (Student 2, Chile).

They also considered didactic innovation, one of the components of the ecological suitability criterion:

I think that the didactic proposals should be striking and innovative that seek or serve as support to the teacher to improve the learning development of students by capturing their attention. (Student 1, Ecuador).

Through new, more innovative methodologies, with concrete, symbolic and pictorial material. Leaving aside the traditional way of teaching mathematics, where it is learned or understood through memorization or meaningless mechanical learning (Student 4, Chile).

There was a future teacher, in particular, who in his statement has implicitly suggested that improving the teaching and learning of mathematics is related to the epistemic part, that is, the knowledge of the mathematical content to be taught; learning, in particular, formative evaluation processes and curricular adaptation and the media, in particular, the time necessary to work on certain content:

Take into account the content. Mastering the subject and content. Adapt to the environment. Analyze the way students learn. Visualize if there are special educational needs and make adaptations for them. Calculate a reasonable time for teaching the subject. Teach the class. Take assessments (ongoing at the end). With respect to evaluations, reinforce the topic taught. (Student 23, Ecuador).

Another relevant aspect emerges from the discourse of future teachers is that improving the teaching and learning of mathematics consists of an improvement in teacher training, in particular, in the strengthening of competence in didactic analysis of their practice:

An improvement in teacher training aimed at finding new paths in the teaching of mathematics in the classroom that it offers. (Student 7, Brazil).

It means seeking strategies and models for constant improvement, focused mainly on teacher training. In order to achieve the benefit for students (Student 5, Chile).

Starting from the observation of a mathematics class given in a certain educational unit, in order to know the advantages and disadvantages of this type of teaching. Carry out an analysis of the aspects that could be improved, put them into practice, and check their effectiveness as a didactic proposal. (Student13, Ecuador).

In quantitative terms, it can be inferred that the majority of the surveyed teachers relate that the improvement of the teaching and learning processes of mathematics are related, to a greater extent to the criteria: cognitive, ecological, and emotional and, to a lesser extent, to the criteria: interactional, mediational and epistemic. That is, in the statements of future teachers, for example, very few comments were made about the improvement being related to the quality of mathematical knowledge that the future teacher should have (epistemic criterion), or to the process of classroom organization and management (interactional criteria). Likewise, they consider it very important to consider the context in which the student operates, didactic innovation (ecological criteria), and prior knowledge and evaluation methods (cognitive criteria).

Perceived meanings that future teachers from Ecuador, Brazil and Chile attribute to MD
MD is perceived to be a technological discipline
MD is perceived to improve the teaching and learning of mathematics
MD is perceived to be a scientific discipline
MD is perceived to be a tool or an innovative didactic in the teaching and learning of mathematics

Table 7: Cross-country analysis of future teachers' perception of MD.

CONCLUSIONS

The objective of this work was to determine the meaning that future teachers attribute to the didactics of mathematics and the criteria that they base the improvement of the teaching and learning process of that discipline. The data allow us to infer that, regardless of the country and the characteristics of the training courses, the meanings attributed to MD are similar. On the one hand, in the three countries, the majority of future teachers consider MD to be a technological discipline whose role is to improve the teaching and learning of mathematics. On the other hand, few future teachers consider MD as a scientific discipline based on theoretical and methodological tools that serve to describe and explain the teaching and learning processes of mathematics. Furthermore, it could be inferred that future teachers present coherent arguments when explaining how the teaching and learning processes of mathematics could be improved. These arguments are mainly related to some current principles and trends in Mathematics Education and some results of research carried out in the field of Mathematics Didactics, as indicated by Breda, Font, and Pino-Fan (2018) and Esque and Breda (2021).

In this line, the present study coincides with the need to respond to the new paradigm of the professor-researcher (Nortes and Nortes, 2011; Oliveira and Fiorentini, 2018; Zumaeta, Fuster and Ocaña, 2018), so that the scientific and technological character of the MD can be recognized

(Godino, Batanero and Font, 2019) in order to problematize their professional work and provide well-founded answers to improve teaching and learning processes.

Although the limitations of this qualitative study are related to a contingent of 49 future teachers, the criteria considered by them to generate an improvement in the teaching and learning processes of mathematics are interlinked. Above all, to the cognitive, ecological, and emotional criteria and, to a lesser extent, to the interactional criteria, mediational and epistemic. This study confirms findings of the previous research (Breda, 2020; Breda, Pino-Fan and Font, 2017; Giacomone, Godino and Beltrán-Pellicer, 2018; Hummes, Font and Breda, 2019; Morales-López and Font, 2017; 2019; Sánchez, Font, and Breda, 2021; Seckel, Breda, Sánchez and Font, 2019; Seckel and Font, 2015; 2020), where the implicit use of the didactical suitability criteria is observed by mathematics teachers (in this case, during their initial training).

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